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A sufficient condition for the existence of a G-compactification

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ABSTRACT

In this paper we show that if a topological transformation group is equicontinuous with respect to some uniformity for the phase space, then the topological transformationgroup can equivariantly be embedded in a topological transformationgroup with compact phase space.

1. INTRODUCTION

Our motivation for this paper comes from a problem which was stated in [6], namely, give sufficient conditions for a continuous action of a topological group G on a Tychonoff space X in order that the action can be extended to a continuous action of G on some compactification of X. Here an *action of G on X* is a continuous mapping π: $G\times X\to X$ such that

(1) $\pi(e,x)=x$ for all $x\in X$ (e denotes the unit of G);

(2) $\pi(s,\pi(t,x))=\pi(st,x)$ for all $s,t\in G$ and $x\in X$.

If we define for every $t\in G$ and $x\in X$ the mappings π^t: $X\to X$ and π_x: $G\to X$ by $\pi^t y := \pi(t,y)$ $(y\in G)$ and $\pi_x s := \pi(s,x)$ $(s\in G)$, then π^t is a homeomorphism of X onto itself, and π_x is continuous. Moreover the mapping $\bar\pi$: $t\mapsto\pi^t$: $G\to\mathscr{H}(X)$ of G into the full homeomorphism group $\mathscr{H}(X)$ of X is a homomorphism.

If π is an action of G on X, then the triple $\langle G,X,\pi\rangle$ will be called a *G-space* or a *topological transformation group* (abbreviation: ttg), and X is called the *phase space* of the ttg. A *G-compactification* of the ttg $\langle G,X,\pi\rangle$ consists of a ttg $\langle G,Y,\sigma\rangle$ with compact Hausdorff phase space Y, together with a dense topological embedding f: $X\to Y$ such that $\sigma^t\circ f=f\circ\pi^t$ for all $t\in G$. So our problem is to find sufficient conditions for a ttg $\langle G,X,\pi\rangle$ to have a G-compactification. Since it is obviously a necessary condition that X is a Tychonoff space, *we shall assume from now on that X is a Tychonoff space* (*i.e. a completely regular T_1-space*).

It is not difficult to find a compactification Y of X such that each π^t can be extended to a homeomorphism σ^t of Y (e.g. take $Y:=\beta X$, the Stone-Čech compactification of X), but the difficulty is to show then that the mapping σ: $(t,y)\mapsto\sigma^t y$: $G\times Y\to Y$ is continuous (in the particular case that we take $Y=\beta X$, σ is usually not continuous; cf. [8], Example 2.3). A straightforward compactness-argument shows, that *if X is locally compact, then $\langle G,X,\pi\rangle$ has a G-compactification*: G acts continuously on the one-point compactification $X\cup\{\infty\}$ of X, leaving ∞ invariant (local compactness of G is not required). In [7] it was shown that $\langle G,X,\pi\rangle$ *has always a G-compactification if the group G is locally compact* (X an arbitrary Tychonoff space). In this note, we show that if X admits a uniformly $\mathscr{U}$ which is compatible with the topology of X such that the so-called *transition group* $\bar\pi[G]=\{\pi^t\,|\,t\in G\}$ is $\mathscr{U}$-equicontinuous on X, then the ttg $\langle G,X,\pi\rangle$ has a G-compactification.

2. PRELIMINARIES

In this section, $\langle G,X,\pi\rangle$ will always denote a ttg with a Tychonoff phase space X. A uniformity which is compatible with the topology of X, i.e. which generates the topology of X, will shortly be called a uniformity *for* X. If $\mathscr{U}$ is a uniformity in X, then the action π and the ttg $\langle G,X,\pi\rangle$ are called *bounded with respect to* $\mathscr{U}$, or *$\mathscr{U}$-bounded*, if, first, $\mathscr{U}$ is a uniformity for X and second, one of the following mutually equivalent conditions is fulfilled:

(1) The family $\{\pi_x\,|\,x\in X\}$ of functions from G to X is equicontinuous at e, that is,

$$\forall\alpha\in\mathscr{U}\,\exists V\in\mathscr{V}_e:\ (x,\pi^t x)\in\alpha \text{ for all } t\in V \text{ and } x\in X;$$

here $\mathscr{V}_e$ denotes the family of all neighbourhoods of e in G;

(2) the mapping $\bar\pi$: $G\to\mathscr{H}_u(X)$ is continuous at e; here $\mathscr{H}_u(X)$ is the full homeomorphism group of X endowed with the topology of $\mathscr{U}$-uniform convergence on X;

(3) if for every $\alpha\in\mathscr{U}$ the subset V_α of G is defined by

$$V_\alpha := \bigcap_{x\in X} V_{x,\alpha}, \text{ where } V_{x,\alpha} := \{t\in G\,|\,(x,\pi^t x)\in\alpha\},$$

then $V_\alpha\in\mathscr{V}_e$ for every $\alpha\in\mathscr{U}$.

The following proposition is fundamental for our result. The implication (ii)$\Rightarrow$(i) generalizes a result of Brook [3].

2.1. PROPOSITION. *The following conditions are equivalent for* $\langle G,X,\pi\rangle$:

(i) $\langle G,X,\pi\rangle$ *has a G-compactification* $\langle G,Y,\sigma\rangle$;

(ii) *there exists a uniformity $\mathscr{U}$ for X such that π is $\mathscr{U}$-bounded.*

PROOF. The proof of (i)$\Rightarrow$(ii) is a straightforward compactness argument: take for $\mathscr{U}$ the relative uniformity of X in Y. The proof of (ii)$\Rightarrow$(i) is, essentially, an application of ASCOLI's theorem to the subset $\{\pi_x | x \in X\}$ of $C(G,X)$, where, in turn, $C(G,X)$ may be considered as a subset of $C(G,[0,1]^\kappa)$ (both function spaces with the compact-open topology), the inclusion of $C(G,X)$ into $C(G,[0,1]^\kappa)$ being induced by a suitable embedding of X in $[0,1]^\kappa$ for some cardinal number κ. For details cf. [6], 7.3.12. □

2.2. EXAMPLE. If G is non-discrete, then the cardinal number $p(G)$ is defined as the least cardinal number of a subset $\mathscr{B}$ of $\mathscr{V}_e$ such that $\bigcap \mathscr{B} \notin \mathscr{V}_e$. Thus, always $p(G) \geq \aleph_0$, and $p(G) > \aleph_0$ if and only if G is a P-group, that is, every G_δ-set in G is open. Now the following is easy to prove: if $\mathscr{U}$ is a uniformity for X, $\alpha \in \mathscr{U}$ and α is *closed* as a subset of $X \times X$, then for every *dense* subset A of X we have $V_\alpha = \bigcap \{V_{x,\alpha} | x \in A\}$. Therefore, if $d(X)$ denotes the *density* of X (i.e. the least cardinal number of a dense subset of X), then $\langle G,X,\pi \rangle$ is $\mathscr{U}$-bounded for every uniformity $\mathscr{U}$ for X if $d(X) < p(G)$.

2.3. COROLLARY. *If $d(X) < p(G)$, then $\langle G,X,\pi \rangle$ has a G-compactification. In particular if G is a P-group and X is separable, then $\langle G,X,\pi \rangle$ has a G-compactification.* □

2.4 REMARK. There exist many P-groups which are non-discrete (hence non-locally compact; cf. [4], Exercise 4K2). So the above corollary is not covered by the result of [7].

3. MAIN RESULT

The proof of our main theorem consists of two steps, which are performed in Lemma 3.1 and Proposition 3.2 below. The idea is to modify a given uniformity so as to make the given action bounded w.r.t. the new uniformity. The first step is due to Anantharaman and Naimpally [1], and the second one is, essentially, the formation of a quotient uniformity, induced by the mapping π: $G \times X \to X$, when $G \times X$ is given the uniformity $\mathscr{R} \times \mathscr{W}$, $\mathscr{R}$ being the right uniformity on G and $\mathscr{W}$ a suitable uniformity for X.

3.1. LEMMA. *If $\mathscr{F}$ is a semigroup of selfmaps of X and $\mathscr{U}$ is a uniformity for X such that $\mathscr{F}$ is $\mathscr{U}$-equicontinuous, then there exists a uniformity $\mathscr{U}^*$ for X such that $\mathscr{F}$ is $\mathscr{U}^*$-uniformly equicontinuous.*

PROOF. Take for $\mathscr{U}^*$ the uniformity, generated by all sets of the form

$$\alpha^* := \bigcap_{f \in \mathscr{F}} \{(x,y) | (x,y) \in X \times X \; \& \; (f(x), f(y)) \in \alpha\} \cap \alpha$$

with $\alpha \in \mathscr{U}$. For further details, cf. [1]. □

3.2. PROPOSITION. *Let $\mathscr{W}$ be a uniformity for X such that $\{\pi^t | t \in G\}$ is $\mathscr{W}$-uniformly equicontinuous. Then there exists a uniformity $\mathscr{W}'$ for X such that $\langle G,X,\pi \rangle$ is $\mathscr{W}'$-bounded.*

PROOF. For $\alpha \in \mathscr{W}$ and $U \in \mathscr{V}_e$, put

$$\begin{aligned}[\alpha, U] &:= \{(\pi^s p, \pi^t q) \mid (p,q) \in \alpha \ \& \ (s,t) \in G \times G \ \& \ st^{-1} \in U\} \\ &= \{(x,y) \in X \times X \mid \exists (u,v) \in G \times G \text{ with } u^{-1}v \in U \text{ and } \\ &\qquad (\pi^u x, \pi^v y) \in \alpha\}.\end{aligned}$$

Then $\mathscr{B}' := \{[\alpha, U] \mid (\alpha, U) \in \mathscr{W} \times \mathscr{V}_e\}$ is a base for a uniformity. Indeed, the diagonal of $X \times X$ is included in $[\alpha, U], [\alpha, U]^{-1} = [\alpha^{-1}, U^{-1}]$ and $[\alpha, U] \cap \cap [\beta, V] \supseteq [\alpha \cap \beta, U \cap V]$ for all $\alpha, \beta \in \mathscr{W}$ and $U, V \in \mathscr{V}_e$. Only the "triangle inequality"

$$\forall [\alpha, U] \in \mathscr{B}' \exists [\alpha_1, U_1] \in \mathscr{B}' : [\alpha_1, U_1] \circ [\alpha_1, U_1] \subseteq [\alpha, U]$$

needs serious checking. First, if $[\alpha, U] \in \mathscr{B}'$ is given, take $\gamma \in \mathscr{W}$ such that $\gamma \circ \gamma \subseteq \alpha$. By $\mathscr{W}$-uniform equicontinuity there exists $\alpha_1 \in \mathscr{W}$ such that for all $x, y \in X$,

$$(x,y) \in \alpha_1 \Rightarrow (\pi^t x, \pi^t y) \in \gamma \text{ for all } t \in G.$$

Next, take $U_1 \in \mathscr{V}_e$ such that $U_1^2 \subseteq U$. Now a rather straightforward computation shows that $[\alpha_1, U_1] \circ [\alpha_1, U_1] \subseteq [\alpha, U]$.

Let $\mathscr{W}'$ be the uniformity generated by $\mathscr{B}'$. If $\mathscr{T}$ is the topology of X (thus, $\mathscr{T}$ is generated by $\mathscr{W}$) and $\mathscr{T}'$ is the topology, generated by $\mathscr{W}'$, then $\mathscr{T}' \subseteq \mathscr{T}$, because $[\alpha, U] \supseteq \alpha$ for every $[\alpha, U] \in \mathscr{B}'$. Conversely, let $U_0 \in \mathscr{T}$ and let $x_0 \in U_0$. As π is continuous, there are $V \in \mathscr{V}_e$ and $\alpha \in \mathscr{W}$ such that for all $t \in G$ and $x \in X$,

$$t \in V^{-1} \ \& \ (x_0, x) \in \alpha \Rightarrow \pi^t x \in U_0.$$

By $\mathscr{W}$-uniform equicontinuity, there exists $\alpha_2 \in \mathscr{W}$ such that for all $x, y \in X$,

$$(x,y) \in \alpha_2 \Rightarrow (\pi^t x, \pi^t y) \in \alpha \text{ for all } t \in G.$$

It follows easily, that $[\alpha_2, V](x_0) = \{x \in X \mid (x_0, x) \in [\alpha_2, V]\} \subseteq U_0$, which shows that $\mathscr{T} \subseteq \mathscr{T}'$. Hence $\mathscr{W}'$ is a uniformity for X.

Finally, if $[\alpha, U] \in \mathscr{B}'$, then for all $x \in X$ and $t \in U$ we have $(x, \pi^t x) \in [\alpha, U]$. This shows, that $\langle G, X, \pi \rangle$ is $\mathscr{W}'$-bounded. □

3.3. THEOREM. *If $\langle G, X, \pi \rangle$ is a ttg with X a Tychonoff space such that the transitiongroup $\{\pi^t \mid t \in G\}$ is $\mathscr{U}$-equicontinuous with respect to some uniformity $\mathscr{U}$ for X, then $\langle G, X, \pi \rangle$ has a G-compactification.*

PROOF. By 3.1 and 3.2, $\langle G, X, \pi \rangle$ is $\mathscr{W}'$-bounded with respect to some uniformity $\mathscr{W}'$ for X. Now apply 2.1. □

3.4. REMARKS. (1) If G is *compact* and $\langle G, X, \pi \rangle$ is a ttg with Tychonoff phase space X, then by a straightforward compactness argument, the continuity of π implies that $\{\pi^t \mid t \in G\}$ is equicontinuous w.r.t. any uniformity $\mathscr{U}$ for X. So $\langle G, X, \pi \rangle$ has a G-compactification. (By completely different arguments, this follows also from [7].)

(2) The following example shows that the uniformities $\mathscr{U}$, $\mathscr{U}^*$ and $\mathscr{U}^{*\prime}$,

formed according to the proofs of 3.1 and 3.2, may be mutually different. In fact, $\mathscr{U}$-equicontinuity does not imply $\mathscr{U}$-uniform equicontinuity, and this, in turn, does not imply $\mathscr{U}'$-boundedness for ttgs, not even if the group G is compact. Let $\mathbb{C}$ be the complex plane and let, for every $n \in \mathbb{N}$, $X_n := \{z \in \mathbb{C} \mid |z| = (2n+1)(n+1)^{-1}\}$ and $X := \bigcup\{X_n \mid n \in \mathbb{N}\}$. Consider the following action π of the torus group $\mathbb{K} := \mathbb{R}/\mathbb{Z}$ (to be identified with the subset $\{t \in \mathbb{C} \mid |t| = 1\}$ of $\mathbb{C}$) on X:

$$\pi(t,z) := t^n z \text{ if } (t,z) \in \mathbb{K} \times X_n.$$

Since $\mathbb{K}$ is compact, the ttg $\langle \mathbb{K}, X, \pi \rangle$ is equicontinuous w.r.t. any uniformity for X. Let $\mathscr{U}$ denote the uniformity for X, induced by the additive uniformity of $\mathbb{C}$ (i.e. the uniformity, induced by the metric $(z_1, z_2) \mapsto |z_1 - z_2|$). Then $\{\pi^t \mid t \in \mathbb{K}\}$ is not $\mathscr{U}$-uniformly equicontinuous, for otherwise the topology of $\mathscr{U}$-uniform convergence on $\{\pi^t \mid t \in \mathbb{K}\}$ would coincide with the topology of pointwise convergence (cf. [2], Chap. X, § 2, Theorem 1). But it is not difficult to show that this is not true. In fact, consider the points $t_n := \exp(i\pi/n)$ for $n \in \mathbb{N}$; then $\bar{\pi}(t_n) \to \bar{\pi}(1)$ pointwise, whereas $\{\bar{\pi}(t_n)\}_{n \in \mathbb{N}}$ cannot converge to $\bar{\pi}(1)$ uniformly. Indeed, for each $n \in \mathbb{N}$ and $z_n \in X_n$ we have $|\pi(t_n, z_n) - z_n| = |-2z_n| > 1$. Consequently, the uniformity $\mathscr{U}^*$, constructed according to 3.1, must be different from $\mathscr{U}$. Moreover, $\langle \mathbb{K}, X, \pi \rangle$ is not $\mathscr{U}$-bounded, because otherwise 2.1 would imply that $\bar{\pi}(t_n) \to \bar{\pi}(1)$ $\mathscr{U}$-uniformly, which we just showed to be not true. Since $\mathscr{U} \subset \mathscr{U}^*$ (see the proof of 3.1) it follows that $\langle \mathbb{K}, X, \pi \rangle$ is not $\mathscr{U}^*$-bounded either. So we must have $\mathscr{U}^{*\prime} \neq \mathscr{U}^*$ and also $\mathscr{U}^{*\prime} \neq \mathscr{U}$.

(3) The "disadvantage" of the result, formulated in Theorem 3.3 is, that a condition about uniformities is needed for a purely topological conclusion. In the next result, this will be avoided. The idea was borrowed from [1].

3.5. COROLLARY. *Let $\langle G, X, \pi \rangle$ be a ttg with Tychonoff phase space X such that the following conditions are fulfilled:*
(1) The transition group $\{\pi^t \mid t \in G\}$ is evenly continuous on X;
(2) for every $x \in X$, the orbit closure $\pi_x[G]$ is compact.
Then $\langle G, X, \pi \rangle$ has a G-compactification.

PROOF. The conditions (1) and (2) imply that the transition group is equicontinuous with respect to some uniformity for X. Cf. [5], p. 237. □

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